

Akashi series, characteristic elements and congruence of Galois representations

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Abstract

In this paper, we compare the Akashi series of the Pontryagin dual of the Selmer groups of two Galois representations over a strongly admissible p -adic Lie extension. Namely, we show that whenever the two Galois representations in question are congruent to each other, the Akashi series of one is a unit if and only if the Akashi series of the other is also a unit. We will also obtain similar results for the Euler characteristics of the Selmer groups and the characteristic elements attached to the Selmer groups.

Keywords and Phrases: Selmer groups, strongly admissible p -adic Lie extensions, Akashi series, Euler characteristics, characteristic elements.

Mathematics Subject Classification 2010: 11R23, 11R34, 11G05, 11F80.

1 Introduction

Let p be an odd prime. In this paper, we are interested in studying the Selmer groups attached to a certain class of Galois representations. This class of Galois representations will contain those which arises from an abelian variety having good ordinary reduction at all primes above p , and those which arises from p -ordinary modular forms. We will be concerned with comparing the Selmer groups of two congruent Galois representations. Over the cyclotomic \mathbb{Z}_p -extension, such studies were carried out in [EPW, G94, GV, Ha]. One of the motivation behind these studies lies in the philosophy that the “Iwasawa main conjecture” should be preserved by congruences. Naturally, one will like to make an analogous study over a noncommutative p -adic Lie extension and this has been carried out in [Ch2, CS, Sh, SS, SS2] to some extent.

The aim of this paper is to compare the Akashi series (and its related invariants) of the Selmer groups of the two congruent Galois representations over a strongly admissible p -adic Lie extension which is not totally real. Our main result is that under appropriate assumptions, whenever the two Galois representations in question are congruent to each other, the Akashi series of the Selmer group of one is a unit if and only if the Akashi series of the other is a unit (see Theorem 5.1). We will also prove similar statements for the Euler characteristics of the Selmer groups (see Theorem 5.4) and for the characteristic elements attached to the Selmer groups (see Theorem 6.3).

We now give a brief description of the idea behind our proofs. One of the main ingredient to our proofs is a result of Matsuno [Mat] which is a fine analysis on the structure of the Selmer groups of the

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Galois representations over a cyclotomic \mathbb{Z}_p -extension of a non-totally real field (see also [Ch]). Building on Matsuno's result, we are able to obtain certain fine results on the structure of the Selmer group over an admissible p -adic Lie extension which is not totally real (see Lemma 4.2). Roughly speaking, the first part of our results will show that the Selmer group, if nonzero, cannot be "too small". Another ingredient we require is an observation on the structure of the homology of the dual Selmer groups by the author in [Lim]. Here, again roughly speaking, our result will yield that if the Akashi series of the Selmer group is a unit, then the Selmer group is "too small" (see Proposition 2.2) and so by the previous observation, the Selmer group has to be zero. Building on this observation (see Section 5), we are reduced to showing that the dual Selmer groups have the same ranks over the Iwasawa algebra of $\text{Gal}(F_\infty/F^{\text{cyc}})$. We will prove this equality of ranks under certain appropriate assumptions (see Proposition 5.3) and hence establish our main results. It is here that we will require the second result of Lemma 4.2 which asserts that the π -torsion submodule of the dual Selmer group is trivial.

We should remark that one can prove the main results in this article under the assumption that the dual Selmer group in question has the property that it does not contain any nontrivial pseudo-null submodule. For the case of dual Selmer groups of elliptic curves, this property has been verified in many cases (see [HO, OcV]). However, for dual Selmer groups of other Galois representations, such pseudo-nullity results are less complete (but see [Oc, Su]). However, if one is willing to work with p -adic extensions that are not totally real, we will see in this paper that we do not require this property. It is also worthwhile mentioning that we do not make use of any explicit formulas for the Akashi series and Euler characteristics, and so our approaches differ from those in [Sh, SS2].

We now give a brief description of the layout of the paper. In Section 2, we recall certain algebraic notion which will be used in the subsequent of the paper. In Section 3, we review the main result of Matsuno [Mat] on the structure of the Selmer groups over the cyclotomic \mathbb{Z}_p -extension. In Section 4, we review some of the results in [Lim] which we require. We will also show how Matsuno's result can be applied to establish certain result on the structure of the Selmer groups which is crucial in the proof of our main results. In Section 5, we will prove our main results on the Akashi series and Euler characteristics. In the final section, we will prove the analogous result for the characteristic elements of the Selmer groups.

2 Algebraic Preliminaries

In this section, we recall some algebraic preliminaries that will be required in the later part of the paper. Let p be an odd prime. Let \mathcal{O} be the ring of integers of a finite extension of \mathbb{Q}_p . Let G be a compact p -adic Lie group without p -torsion. The completed group algebra of G over \mathcal{O} is given by

$$\mathcal{O}[[G]] = \varprojlim_U \mathcal{O}[G/U],$$

where U runs over the open normal subgroups of G and the inverse limit is taken with respect to the canonical projection maps. It is well known that $\mathcal{O}[[G]]$ is an Auslander regular ring (cf. [V, Theorem 3.26]). If G is pro- p , then $\mathcal{O}[[G]]$ has no zero divisors (cf. [Neu]). Therefore, $\mathcal{O}[[G]]$ admits a skew field $K(G)$ which is flat over $\mathcal{O}[[G]]$ (see [GW, Chapters 6 and 10] or [Lam, Chapter 4, §9 and §10]). If M is a finitely generated $\mathcal{O}[[G]]$ -module, we define the $\mathcal{O}[[G]]$ -rank of M to be

$$\text{rank}_{\mathcal{O}[[G]]}(M) = \dim_{K(G)} (K(G) \otimes_{\mathcal{O}[[G]]} M).$$

We say that the module M is a *torsion* $\mathcal{O}[[G]]$ -module if $\text{rank}_{\mathcal{O}[[G]]} M = 0$.

We continue to assume that G is pro- p . Fix a local parameter π for \mathcal{O} and denote the residue field of \mathcal{O} by k . The completed group algebra of G over k is defined similarly as above. For a $k[[G]]$ -module N , we then define its $k[[G]]$ -rank by

$$\text{rank}_{k[[G]]}(N) = \frac{\text{rank}_{k[[G_0]]}(N)}{|G : G_0|},$$

where G_0 is an open normal uniform pro- p subgroup of G . This is integral and independent of the choice of G_0 (see [Ho, Proposition 1.6]). Similarly, we will say that that N is a *torsion* $\mathbb{F}_p[[G]]$ -module if $\text{rank}_{\mathbb{F}_p[[G]]} N = 0$.

For a given finitely generated $\mathcal{O}[[G]]$ -module M , we denote $M(\pi)$ to be the $\mathcal{O}[[G]]$ -submodule of M consisting of elements of M which are annihilated by some power of π . Since the ring $\mathcal{O}[[G]]$ is Noetherian, the module $M(\pi)$ is finitely generated over $\mathcal{O}[[G]]$. Therefore, one can find an integer $r \geq 0$ such that π^r annihilates $M(\pi)$. Following [Ho, Formula (33)], we define

$$\mu_{\mathcal{O}[[G]]}(M) = \sum_{i \geq 0} \text{rank}_{k[[G]]} (\pi^i M(\pi) / \pi^{i+1}).$$

(For another alternative, but equivalent, definition, see [V, Definition 3.32].) By the above discussion, the sum on the right is a finite one.

We introduce another algebraic invariant which was first defined in [CSS]. As before, G is a compact (not necessarily pro- p) p -adic Lie group without p -torsion. Let H be a closed normal subgroup of G with $\Gamma := G/H \cong \mathbb{Z}_p$. We say that the *Akashi series* of M exists if $H_i(H, M)$ is $\mathcal{O}[[\Gamma]]$ -torsion for every i . In the case of this event, we denote $\text{Ak}_H(M)$ to be the *Akashi series* of M which is defined to be

$$\prod_{i \geq 0} g_i^{(-1)^i},$$

where g_i is the characteristic polynomial of $H_i(H, M)$. Of course, the Akashi series is only well-defined up to a unit in $\mathcal{O}[[\Gamma]]$. Also, note that since G (and hence H) has no p -torsion, H has finite p -cohomological dimension, and therefore, the alternating product is a finite product. For the remainder of the section, identify $\mathcal{O}[[\Gamma]] \cong \mathcal{O}[[T]]$ under a choice of a generator of Γ . A polynomial $T^n + c_{n-1}T^{n-1} + \cdots + c_0$ in $\mathcal{O}[T]$ is said to be a *Weierstrass polynomial* if π divides c_i for every $0 \leq i \leq n-1$. We now record the following proposition which gives a relation between the Akashi series and various Iwasawa invariants when G is pro- p .

Lemma 2.1. *Let G be a compact pro- p p -adic group without p -torsion, and let H be a closed normal subgroup of G with $\Gamma := G/H \cong \mathbb{Z}_p$. Let M be a finitely generated torsion $\mathcal{O}[[G]]$ -module which satisfies the following properties:*

- (i) $H_0(H, M)$ is a finitely generated torsion $\mathcal{O}[[G]]$ -module.
- (ii) $H_i(H, M)$ is finitely generated over \mathcal{O} for every $i \geq 1$.

Then we have

$$\text{Ak}_H(M) = \pi^{\mu_{\mathcal{O}[[\Gamma]]}(H_0(H, M))} \frac{f}{g},$$

where f and g are Weierstrass polynomials.

Proof. This is an easy exercise which we leave to the reader. \square

We will also record the following proposition which is a special case of [Lim, Proposition 5.4].

Proposition 2.2. *Let G be a compact pro- p p -adic group without p -torsion, and let H be a closed normal subgroup of G with $\Gamma := G/H \cong \mathbb{Z}_p$. Let M be a finitely generated $\mathcal{O}[[G]]$ -module which satisfies all of the following properties.*

- (i) $H_0(H, M)$ is a finitely generated torsion $\mathcal{O}[[G]]$ -module.
- (ii) $H_i(H, M)$ is finitely generated over \mathcal{O} for every $i \geq 1$.
- (iii) $Ak_H(M)$ lies in $\mathcal{O}[[\Gamma]]^\times$.

Then M is a finitely generated torsion $\mathcal{O}[[H]]$ -module.

We say that the *G -Euler characteristics* of an $\mathcal{O}[[G]]$ -module M exists if $H_i(G, M)$ is finite for each $i \geq 0$. In the event of such, the *G -Euler characteristics* is given by

$$\chi(G, M) = \prod_{i \geq 0} |H_i(G, M)|^{(-1)^i}.$$

Again, since G has no p -torsion, the above product is a finite one. The connection between the Akashi series and the Euler characteristics is given as follow.

Proposition 2.3. *Let G be a compact p -adic group without p -torsion, and let H be a closed normal subgroup of G with $G/H \cong \mathbb{Z}_p$. Let M be a finitely generated $\mathcal{O}[[G]]$ -module whose G -Euler characteristics is well defined. Then the Akashi series of M exists and we have*

$$\chi(G, M) = |\varphi(Ak_H(M))|_\pi^{-1},$$

where $|\cdot|_\pi$ is the π -adic norm with $|\pi|_\pi = q^{-1}$ (here q is the order of k), and φ is the augmentation map from $\mathcal{O}[[\Gamma]]$ to \mathcal{O} .

Proof. See [CFKSV, Theorem 3.6] or [CSS, Lemma 4.2]. \square

3 Selmer groups over cyclotomic \mathbb{Z}_p -extension

As before, p will denote an odd prime and \mathcal{O} will denote the ring of integers of a fixed finite extension K of \mathbb{Q}_p . Suppose that we are given the following datum $(A, \{A_v\}_{v|p})$ defined over a number field F :

- (C1) A is a cofinitely generated cofree \mathcal{O} -module of \mathcal{O} -corank d with a continuous, \mathcal{O} -linear $\text{Gal}(\bar{F}/F)$ -action which is unramified outside a finite set of primes of F .
- (C2) For each prime v of F above p , A_v is a $\text{Gal}(\bar{F}_v/F_v)$ -submodule of A which is cofree of \mathcal{O} -corank d_v .
- (C3) For each real prime v of F , we write $A_v^+ = A^{\text{Gal}(\bar{F}_v/F_v)}$ which is assumed to be cofree of \mathcal{O} -corank d_v^+ .

(C4) The following equality

$$\sum_{v|p} (d - d_v)[F_v : \mathbb{Q}_p] = dr_2(F) + \sum_{v \text{ real}} (d - d_v^+) \quad (1)$$

holds. Here $r_2(F)$ denotes the number of complex primes of F .

We now consider the base change property of our datum. Let L be a finite extension of F . We can then obtain another set of datum $(A, \{A_w\}_{w|p})$ over L as follows: we consider A as a $\text{Gal}(\bar{F}/L)$ -module, and for each prime w of L above p , we set $A_w = A_v$, where v is a prime of F below w , and view it as a $\text{Gal}(\bar{F}_v/L_w)$ -module. Then $d_w = d_v$. For each real prime w of L , one sets $A^{\text{Gal}(\bar{L}_w/L_w)} = A^{\text{Gal}(\bar{F}_v/F_v)}$ and writes $d_w^+ = d_v^+$, where v is a real prime of F below w . In general, the d_w 's and d_w^+ 's need not satisfy equality (C4). We now record the following lemma which gives some sufficient conditions for the equality in (C4) to hold for the datum $(A, \{A_w\}_{w|p})$ over L (see [LimMu, Lemma 4.1] for the proofs).

Lemma 3.1. *Suppose that $(A, \{A_v\}_{v|p})$ is a datum defined over F which satisfies (C1) – (C4). Suppose further that at least one of the following statements holds.*

- (i) *All the archimedean primes of F are unramified in L .*
- (ii) *$[L : F]$ is odd*
- (iii) *F is totally imaginary.*
- (iv) *F is totally real, L is totally imaginary and*

$$\sum_{v \text{ real}} d_v^+ = d[F : Q]/2.$$

Then we have the equality

$$\sum_{w|p} (d - d_w)[L_w : \mathbb{Q}_p] = dr_2(L) + \sum_{w \text{ real}} (d - d_w^+).$$

We now introduce the Selmer groups. Let S be a finite set of primes of F which contains all the primes above p , the ramified primes of A and all infinite primes. Denote F_S to be the maximal algebraic extension of F unramified outside S and write $G_S(\mathcal{L}) = \text{Gal}(F_S/\mathcal{L})$ for every algebraic extension \mathcal{L} of F which is contained in F_S . Let L be a finite extension of F contained in F_S such that the datum $(A, \{A_w\}_{w|p})$ satisfies (C1)–(C4). For a prime w of L lying over S , set

$$H_{\text{str}}^1(L_w, A) = \begin{cases} \ker(H^1(L_w, A) \rightarrow H^1(L_w, A/A_w)) & \text{if } w \text{ divides } p, \\ \ker(H^1(L_w, A) \rightarrow H^1(L_w^{\text{ur}}, A)) & \text{if } w \text{ does not divide } p, \end{cases}$$

where L_w^{ur} is the maximal unramified extension of L_w . The (strict) Selmer group attached to the datum is then defined by

$$\text{Sel}^{\text{str}}(A/L) := \ker \left(H^1(G_S(L), A) \rightarrow \bigoplus_{w \in S_L} \frac{H^1(L_w, A)}{H_{\text{str}}^1(L_w, A)_{\text{div}}} \right).$$

Here N_{div} denotes the maximal \mathcal{O} -divisible submodule of N and S_L denotes the set of primes of L above S . We now denote F^{cyc} to be the cyclotomic \mathbb{Z}_p -extension of F , and F_n its intermediate subextension with $[F_n : F] = p^n$. Write $\Gamma = \text{Gal}(F_\infty/F)$ and $\Gamma_n = \text{Gal}(F_\infty/F_n)$. We define the Selmer group over F^{cyc} by

$$\text{Sel}^{str}(A/F^{cyc}) = \varinjlim_n \text{Sel}^{str}(A/F_n),$$

where the limit runs over the intermediate extensions F_n of F contained in F^{cyc} . We will write $X(A/F^{cyc})$ for the Pontryagin dual of $\text{Sel}^{str}(A/F^{cyc})$.

We now introduce two additional conditions on our datum.

(Fin _{p}) For every prime w of F^{cyc} above p , $H^0(F_w^{cyc}, A_v)$ is finite. Here v is a prime of F under w .

(Fin) $H^0(F^{cyc}, A^*)$ is finite. Here $A^* = \text{Hom}_{\text{cts}}(T_\pi(A), \mu_{p^\infty})$, where $T_\pi(A) = \varprojlim_i A[\pi^i]$.

The following lemma gives an alternative description of the Selmer group over F^{cyc} .

Lemma 3.2. *Suppose that (Fin _{p}) holds. Then we have*

$$\text{Sel}^{str}(A/F^{cyc}) = \ker \left(H^1(G_S(F^{cyc}), A) \longrightarrow \bigoplus_{v \in S} J_v(A/F^{cyc}) \right),$$

where

$$J_v(A/F^{cyc}) = \begin{cases} \bigoplus_{w|v} H^1(F_w^{cyc}, A/A_v), & \text{if } v \text{ divides } p \\ \bigoplus_{w|v} H^1(F_w^{cyc}, A), & \text{if } v \text{ does not divide } p \end{cases}$$

Proof. Since (Fin _{p}) holds, it follows from the discussion in [G89, P 111] that $H^1(F_w^{cyc}, A_v)$ is divisible for $w|p$. One can then proceed using a similar argument to that in [Mat, Lemma 3.2]. \square

For the remainder of the section, we will review the result of Matsuno on the structure of the dual Selmer group which will play an important role in proving the main results of the paper. Now following [Mat], we define the following finite $\mathcal{O}[[\Gamma]]$ -module.

Definition 3.3. For each prime w of F^{cyc} , we define a subgroup of $H^0(F_w^{cyc}, A^*)$ by

$$B_w = \begin{cases} H^0(F_w^{cyc}, A_v^*), & \text{if } w \text{ lies above } v|p, \\ H^0(F_w^{cyc}, A_v^*), & \text{if } w \text{ is archimedean,} \\ H^0(F_w^{cyc}, A_v^*)_{div}, & \text{otherwise.} \end{cases}$$

Then we define a subgroup $B(F^{cyc}, A^*)$ of $H^0(F^{cyc}, A^*)$ by

$$B(F^{cyc}, A^*) = \bigcap_w i_w^{-1}(B_w),$$

where w runs over all primes of F^{cyc} and i_w is the canonical injection

$$H^0(F^{cyc}, A^*) \hookrightarrow H^0(F_w^{cyc}, A^*).$$

For a finitely generated torsion $\mathcal{O}[[\Gamma]]$ -module M , we denote $\text{Fin}_{\mathcal{O}[[\Gamma]]}(M)$ to be the maximal finite submodule of M . (This is well-defined; for instance, see [Ch, Lemma 3.2].) The following lemma gives a relationship between $B(F^{cyc}, A^*)$ and the maximal finite submodule of $X(A/F^{cyc})$.

Lemma 3.4. *Suppose that (Fin_p) and (Fin) hold. Assume that $X(A/F^{\text{cyc}})$ is a torsion $\mathcal{O}[[G]]$ -module. Then there is an $\mathcal{O}[[\Gamma]]$ injection*

$$Fin_{\mathcal{O}[[\Gamma]]} X(A/F^{\text{cyc}}) \hookrightarrow B(F^{\text{cyc}}, A^*).$$

Proof. See [Ch, Proposition 3.9] or [Mat, Corollary 4.3]. \square

Proposition 3.5. *Let p be an odd prime. Assume that (Fin_p) and (Fin) hold. Suppose that F is not totally real and that $X(A/F^{\text{cyc}})$ has a nontrivial finite $\mathcal{O}[[\Gamma]]$ -module. Then $X(A/F^{\text{cyc}})$ is not finitely generated over \mathcal{O} .*

Proof. This is proven by a similar argument to that in [Ch, Theorem 3.11] or [Mat, Proposition 7.5] by appealing to the preceding lemma and the fact that the Galois group of the maximal abelian pro- p extension of F^{cyc} unramified outside p has positive $\mathbb{Z}_p[[\Gamma]]$ -rank when F is not totally real. \square

We record a corollary which will play a crucial role in the subsequent of the paper.

Corollary 3.6. *Let p be an odd prime. Assume that (Fin_p) and (Fin) hold. Suppose that F is not totally real and that $X(A/F^{\text{cyc}})$ is finitely generated over \mathcal{O} . Then $X(A/F^{\text{cyc}})$ is \mathcal{O} -torsionfree.*

We end the section mentioning two basic examples of our datum.

(i) $A = \mathcal{A}[p^\infty]$, where \mathcal{A} is an abelian variety defined over an arbitrary finite extension F of \mathbb{Q} with good ordinary reduction at all places v of F dividing p . For each $v|p$, it follows from [CG, P. 150-151] that we have a $\text{Gal}(\bar{F}_v/F_v)$ -submodule A_v which can be characterized by the property that A/A_v is the maximal $\text{Gal}(\bar{F}_v/F_v)$ -quotient of $\mathcal{A}[p^\infty]$ on which some subgroup of finite index in the inertia group I_v acts trivially. It is not difficult to verify that (C1)-(C4) are satisfied. The condition (Fin_p) is also known to be satisfied (see [Mat, Section 5]) and the condition (Fin) is a well-known consequence of a theorem of Imai [Im].

(ii) Let V be the Galois representation attached to a primitive Hecke eigenform f for GL_2/\mathbb{Q} , which is ordinary at p , relative to some fixed embedding of the algebraic closure of \mathbb{Q} into \mathbb{Q}_p . By the work of Mazur-Wiles [MW], V contains a one-dimensional \mathbb{Q}_p -subspace V_v invariant under $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ with the property that the inertial subgroup I_p acts via a power of the cyclotomic character on V_v and trivially on V/V_v . By compactness, V will always contain a free \mathcal{O} -submodule T , which is stable under the action of $\text{Gal}(\bar{F}/F)$. For such an \mathcal{O} -lattice T , we write $A = V/T$ and $A_v = V_v/(T \cap V_v)$. The condition (Fin_p) can be easily seen to follow from the property that the inertial subgroup I_p acts via a power of the cyclotomic character on V_v (or see the proof of [Sh, Lemma 3.6]). The condition (Fin) is shown in the proof of [Su, Lemma 2.2].

4 Selmer groups over strongly admissible p -adic Lie extensions

We say that F_∞ is a *strongly admissible p -adic Lie extension* of F if (i) $\text{Gal}(F_\infty/F)$ is a compact p -adic Lie group without p -torsion, (ii) F_∞ contains the cyclotomic \mathbb{Z}_p -extension F^{cyc} of F and (iii) F_∞ is unramified outside a finite set of primes of F . We will write $G = \text{Gal}(F_\infty/F)$, $H = \text{Gal}(F_\infty/F^{\text{cyc}})$ and $\Gamma = \text{Gal}(F^{\text{cyc}}/F)$.

Let $(A, \{A_v\}_{v|p})$ be a datum defined as in the previous section which satisfies (C1)–(C4). For the remainder of the paper, we will always assume that for every finite extension L of F in F_∞ , the datum $(A, \{A_w\}_{w|p})$ over L obtained by base change satisfies (C1)–(C4). Let S be a finite set of primes which contains all the primes above p , the ramified primes of A , the archimedean primes and the primes that are ramified in F_∞/F . The Selmer group of the data over F_∞ is defined by

$$\mathrm{Sel}^{str}(A/F_\infty) = \varinjlim_L \mathrm{Sel}^{str}(A/L),$$

where the limit runs over all the finite extensions L of F contained in F_∞ . By a standard argument as in [CS, Section 2], one can show that the Selmer group is independent of the choice of S as long as it contains all the primes above p , the ramified primes of A , the archimedean primes and the primes that are ramified in F_∞/F . We write $X(A/F_\infty)$ for the Pontryagin dual of $\mathrm{Sel}^{str}(A/F_\infty)$. We record the following proposition.

Proposition 4.1. *Suppose that (Fin_p) holds for every finite extension L of F contained in F_∞ . Then we have the following description of the strict Selmer groups.*

$$\mathrm{Sel}^{str}(A/F_\infty) = \ker \left(H^1(G_S(F_\infty), A) \xrightarrow{\lambda_{A/F_\infty}} \bigoplus_{v \in S} J_v(A/F_\infty) \right),$$

where

$$J_v(A/F_\infty) = \varinjlim_L J_v(A/L^{\mathrm{cyc}}).$$

Here the limit is taken over all the finite extensions L of F contained in F_∞ .

Proof. For each L , we denote L_n to be the intermediate extension of L contained in L^{cyc} with $[L : L_n] = p^n$. Then we have

$$\mathrm{Sel}^{str}(A/F_\infty) = \varinjlim_L \varinjlim_n \mathrm{Sel}^{str}(A/L_n) = \varinjlim_L \mathrm{Sel}^{str}(A/L^{\mathrm{cyc}}).$$

The conclusion of the proposition is now immediate from Lemma 3.2. \square

For the remainder of the paper, we will also impose the following condition.

(Fin _{∞}): The data $(A, \{A_v\}_{v|p})$ satisfies (Fin_p) and (Fin) for every finite extension L of F contained in F_∞ .

Lemma 4.2. *Assume that (Fin_∞) is valid, and assume that F_∞ is not totally real. Then the following statements hold.*

- (a) *If $X(A/F_\infty) \neq 0$, then $X(A/F_\infty)$ is not a finitely generated torsion $\mathcal{O}[[H]]$ -module.*
- (b) *If $X(A/F_\infty)$ is finitely generated over $\mathcal{O}[[H]]$, then $X(A/F_\infty)(\pi) = 0$.*

Proof. (a) This is proven by a similar argument to that in [Lim, Lemma 5.8] by appealing to Corollary 3.6.

(b) Suppose that $X(A/F_\infty)$ is finitely generated over $\mathcal{O}[[H]]$. In particular, this implies that $X(A/L^{\mathrm{cyc}})$ is finitely generated over \mathcal{O} for every finite extension L of F contained in F_∞ such that $\mathrm{Gal}(F_\infty/L)$ is

pro- p . Since F_∞ is not totally real, we can rewrite $X(A/F_\infty) = \varprojlim_L X(A/L^{\text{cyc}})$, where L runs through all finite extensions of F contained in F_∞ such that $\text{Gal}(F_\infty/L)$ is pro- p and L is not totally real. By Corollary 3.6, the multiplication by π -map on $X(A/L^{\text{cyc}})$ is an injection. Since inverse limit is left exact, it follows that the multiplication by π -map on $X(A/F_\infty)$ is also an injection. \square

We will write λ_{A/F_∞} for the localization map

$$H^1(G_S(F_\infty), A) \longrightarrow \bigoplus_{v \in S} J_v(A/F_\infty)$$

as given in Proposition 4.1. We can now record the following result.

Proposition 4.3. *Assume that (i) (Fin_∞) holds, (ii) $G = \text{Gal}(F_\infty/F)$ is pro- p and has no p -torsion, (iii) $X(A/F^{\text{cyc}})$ is $\mathcal{O}[[\Gamma]]$ -torsion, (iv) $H^2(G_S(F_\infty), A) = 0$ and (v) λ_{A/F_∞} is surjective. Then the following statements hold.*

- (a) $H_0(H, X(A/F_\infty))$ is $\mathcal{O}[[\Gamma]]$ -torsion and its $\mu_{\mathcal{O}[[\Gamma]]}$ -invariant is precisely $\mu_{\mathcal{O}[[\Gamma]]}(X(A/F^{\text{cyc}}))$.
- (b) $H_i(H, X(A/F_\infty))$ is finitely generated over \mathcal{O} for every $i \geq 1$.
- (c) $H_i(H, X(A/F_\infty)) = 0$ for $i \geq \dim H$.

In particular, we have

$$Ak_H(X(A/F_\infty)) = \pi^{\mu_{\mathcal{O}[[\Gamma]]}(X(A/F^{\text{cyc}}))} \frac{f}{g}$$

for some Weierstrass polynomials f and g .

Proof. The assertions on the H -homology of $X(A/F_\infty)$ can be proven by a similar argument to that in [Lim, Proposition 5.1]. The assertion on the Akashi series is then an immediate consequence of this and Lemma 2.1. \square

Remark 4.4. It follows from a standard argument that conditions (iv) and (v) will imply that $X(A/F_\infty)$ is a torsion $\mathcal{O}[[G]]$ -module. Also, there are many cases of conditions (iv) and (v) following from (iii) and we mention a few of them here.

(a) Suppose G has dimension 2. By our (Fin) assumption, we may apply a standard argument (for instance, see [Lim, Proposition 3.3(a)]) to show that $H^2(G_S(F^{\text{cyc}}), A) = 0$ and $\lambda_{A/F^{\text{cyc}}}$ is surjective. One can now apply an argument similar to [SS, Theorem 7.4] to obtain the conclusion that $H^2(G_S(F_\infty), A) = 0$ and λ_{A/F_∞} is surjective.

(b) G is a solvable uniform pro- p group and $A(F_\infty)$ is finite. Then by a similar argument to that in [HO, Theorem 2.3], we have that $X(A/F_\infty)$ is a torsion $\mathcal{O}[[G]]$ -module. By a standard argument (for instances, see [Lim, Proposition 3.3(a)]), we obtain the conclusion that $H^2(G_S(F_\infty), A) = 0$ and λ_{A/F_∞} is surjective.

(c) G is a solvable uniform pro- p group, and for each $v \in S$, the decomposition group of $G = \text{Gal}(F_\infty/F)$ at v has dimension ≥ 2 . Again by the argument in [HO, Theorem 2.3], we have that $X(A/F_\infty)$ is a torsion $\mathcal{O}[[G]]$ -module. The required conclusion will now follow from the argument in [Lim, Proposition 3.3(b)].

(d) $X(A/F^{\text{cyc}})$ is finitely generated over \mathcal{O} . Then for every finite extension L of F in F_∞ , it follows from the argument in [HM] that $X(A/L^{\text{cyc}})$ is finitely generated over \mathcal{O} , and in particular, torsion over $\mathcal{O}[[\Gamma_L]]$, where $\Gamma_L = \text{Gal}(L^{\text{cyc}}/L)$. Combining this with our (Fin) assumption, we can apply a similar argument to that in [Lim, Proposition 3.3(i), Corollary 3.4] to conclude that $H^2(G_S(F_\infty), A) = 0$ and λ_{A/F_∞} is surjective.

5 Comparing Akashi series of Selmer groups

Retain the notation from the previous section. To state the main result of this section, we need to introduce another datum $(B, \{B_v\}_{v|p})$, where we assume to satisfy conditions (C1)-(C4). We will also assume that the datum obtained by base change over every finite extension of L of F in F_∞ also satisfy conditions (C1)-(C4). From now on, S will always denote a finite set of primes which contains all the primes above p , the ramified primes of A and B , the ramified primes of F_∞/F and the archimedean primes. We introduce the following important congruence condition on A and B which allows us to be able to compare the Selmer groups of A and B .

(Cong) : There is an isomorphism $A[\pi] \cong B[\pi]$ of $G_S(F)$ -modules which induces a $\text{Gal}(\bar{F}_v/F_v)$ -isomorphism $A_v[\pi] \cong B_v[\pi]$ for every $v|p$.

We introduce some notation which we need in our discussion. We write $C_v = A$ for $v \nmid p$ and $C_v = A/A_v$ for $v|p$. Similarly, we write $D_v = B$ for $v \nmid p$ and $D_v = B/B_v$ for $v|p$. Let S_2 denote the set of primes in S such that the decomposition group of $G = \text{Gal}(F_\infty/F)$ at v has dimension ≥ 2 . For every prime w of F^{cyc} above v , we write $C_v(F_w^{\text{cyc}}) = (C_v)^{\text{Gal}(\bar{F}_v/F_w^{\text{cyc}})}$ and $D_v(F_w^{\text{cyc}}) = (D_v)^{\text{Gal}(\bar{F}_v/F_w^{\text{cyc}})}$.

Theorem 5.1. *Let F_∞ be a strongly admissible pro- p p -adic Lie extension of F of dimension at least 2. Assume that F is not totally real. Suppose that all the hypotheses in Proposition 4.3 are valid for A and B , and that **(Cong)** is satisfied. Furthermore, suppose that for every $v \notin S_2$, we have*

$$\dim_k (C_v(F_w^{\text{cyc}})/\pi) = \dim_k (D_v(F_w^{\text{cyc}})/\pi)$$

for every prime w of F^{cyc} above v .

Then $\text{Ak}_H(X(A/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$ if and only if $\text{Ak}_H(X(B/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$.

Remark 5.2. We mention some cases when the final hypothesis of the theorem is satisfied.

(a) $C_v(F_w^{\text{cyc}})$ and $D_v(F_w^{\text{cyc}})$ are \mathcal{O} -divisible. This is immediate. We also note that this has been known to hold in certain cases (see [EPW, Lemma 4.1.3]; also see the proofs of [Sh, Theorem 4.7] and [SS2, Theorem 3.4]).

(b) $C_v(F_w^{\text{cyc}})$ and $D_v(F_w^{\text{cyc}})$ are finite. Then by Howson's formula [Ho, Corollary 1.10], we have

$$\dim_k (C_v(F_w^{\text{cyc}})/\pi) = \dim_k (C_v(F_w^{\text{cyc}})[\pi]).$$

Similarly, we have an analogue formula for D . By the **(Cong)** assumption, we have

$$\dim_k (C_v(F_w^{\text{cyc}})[\pi]) = \dim_k (D_v(F_w^{\text{cyc}})[\pi]).$$

Combining all these equalities, we obtain the required conclusion.

In preparation of the proof of the theorem, we record a proposition.

Proposition 5.3. *Retain the assumptions of Theorem 5.1, then $X(A/F_\infty)$ is finitely generated over $\mathcal{O}[[H]]$ if and only if $X(B/F_\infty)$ is finitely generated over $\mathcal{O}[[H]]$. Furthermore, we have*

$$\text{rank}_{\mathcal{O}[[H]]}(X(A/F_\infty)) = \text{rank}_{\mathcal{O}[[H]]}(X(B/F_\infty)).$$

We now prove Theorem 5.1 assuming the validity of Proposition 5.3.

Proof of Theorem 5.1. Suppose that $Ak_H(X(A/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$. It then follows from Propositions 2.2 and 4.3 that $X(A/F_\infty)$ is a finitely generated torsion $\mathcal{O}[[H]]$ -module. By Proposition 5.3, this in turn implies that $X(B/F_\infty)$ is a finitely generated torsion $\mathcal{O}[[H]]$ -module. Since F is not totally real, it follows from Lemma 4.2(a) that $X(B/F_\infty) = 0$. In particular, this implies that $Ak_H(X(B/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$. \square

The remainder of the section will be devoted to the proof of Proposition 5.3. In preparation of the proof, we first introduce the “mod π ” Selmer group. For every finite extension \mathcal{F} of F^{cyc} , we define $J_v(A[\pi]/\mathcal{F})$ to be

$$\bigoplus_{w|v} H^1(\mathcal{F}_w, A[\pi]) \text{ or } \bigoplus_{w|v} H^1(\mathcal{F}_w, A/A_v[\pi])$$

according as v does not or does divide p . We then define

$$J_v(A[\pi]/F_\infty) = \varinjlim_{\mathcal{F}} J_v(A[\pi]/\mathcal{F}),$$

where the direct limit is taken over all finite extensions \mathcal{F} of F^{cyc} contained in F_∞ . The mod π Selmer group is then defined by

$$\text{Sel}^{\text{str}}(A[\pi]/F_\infty) = \ker \left(H^1(G_S(F_\infty), A[\pi]) \longrightarrow \bigoplus_{v \in S} J_v(A[\pi]/F_\infty) \right).$$

We write $X(A[\pi]/F_\infty)$ for the Pontryagin dual of $\text{Sel}^{\text{str}}(A[\pi]/F_\infty)$. We are now in the position to prove Proposition 5.3.

Proof of Proposition 5.3. Consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Sel}^{\text{str}}(A[\pi]/F_\infty) & \longrightarrow & H^1(G_S(F_\infty), A[\pi]) & \xrightarrow{\psi_A} & \bigoplus_{v \in S} J_v(A[\pi]/F_\infty) \\ & & a \downarrow & & b \downarrow & & c \downarrow \\ 0 & \longrightarrow & \text{Sel}^{\text{str}}(A/F_\infty)[\pi] & \longrightarrow & H^1(G_S(F_\infty), A)[\pi] & \longrightarrow & \bigoplus_{v \in S} J_v(A/F_\infty)[\pi] \end{array}$$

with exact rows. It follows from a standard argument that $\ker a$ and $\text{coker } a$ are cofinitely generated over $k[[H]]$. Hence it follows that $X(A/F_\infty)$ is finitely generated over $\mathcal{O}[[H]]$ if and only if $X(A[\pi]/F_\infty)$ is finitely generated over $k[[H]]$. One also has a similar assertion for B . Now by the **(Cong)** assumption, we have $X(A[\pi]/F_\infty) \cong X(B[\pi]/F_\infty)$. Therefore, the first assertion of the proposition is established.

We proceed to show the second assertion. By Lemma 4.2(b), we have $X(A/F_\infty)[\pi] = 0$, or equivalently, $\text{Sel}^{\text{str}}(A/F_\infty)/\pi = 0$. Therefore, the above diagram can be completed to the following.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Sel}^{str}(A[\pi]/F_\infty) & \longrightarrow & H^1(G_S(F_\infty), A[\pi]) & \xrightarrow{\psi_A} & \bigoplus_{v \in S} J_v(A[\pi]/F_\infty) \longrightarrow \mathrm{coker} \psi_A \longrightarrow 0 \\
& & a \downarrow & & b \downarrow & & c \downarrow \\
0 & \longrightarrow & \mathrm{Sel}^{str}(A/F_\infty)[\pi] & \longrightarrow & H^1(G_S(F_\infty), A)[\pi] & \longrightarrow & \bigoplus_{v \in S} J_v(A/F_\infty)[\pi] \longrightarrow 0
\end{array}$$

Since $\ker b$ and $\ker c$ are cofinitely generated over $k[[H]]$, and b and c are surjective, it follows from a simple diagram chasing argument that $\mathrm{coker} \psi_A$ is cofinitely generated over $k[[H]]$. Furthermore, $\ker b \cong A(F_\infty)/\pi$ is cofinitely generated over k , and hence is a cotorsion $k[[H]]$ -module. Therefore, by another diagram chasing argument, we have

$$\mathrm{corank}_{k[[H]]}(\mathrm{Sel}^{str}(A/F_\infty)[\pi]) = \mathrm{corank}_{k[[H]]}(\ker c) + \mathrm{corank}_{k[[H]]}(\mathrm{Sel}^{str}(A[\pi]/F_\infty)) - \mathrm{corank}_{k[[H]]}(\mathrm{coker} \psi_A).$$

Again, since $\mathrm{Sel}^{str}(A/F_\infty)/\pi = 0$, it follows from [Ho, Corollary 1.10] that

$$\mathrm{corank}_{\mathcal{O}[[H]]}(\mathrm{Sel}^{str}(A/F_\infty)) = \mathrm{corank}_{k[[H]]}(\mathrm{Sel}^{str}(A/F_\infty)[\pi]).$$

We now compute $\mathrm{corank}_{k[[H]]}(\ker c)$. We write $c = \bigoplus_w c_w$, where w runs over the set of primes of F^{cyc} above S . Denote H_w to be the decomposition group of $F_\infty/F^{\mathrm{cyc}}$ corresponding to a fixed prime of F_∞ above w . Then we have $\mathrm{coker} c_w = \mathrm{Coind}_{H_w}^{H_w}(C_v(F_{\infty,w})/\pi)$, where v is the prime of F below w . Now it is easy to verify that

$$\mathrm{corank}_{k[[H]]}(\ker c_w) = \begin{cases} \mathrm{rank}_k(C_v(F_{\infty,w})/\pi) & \text{if } v \notin S_2, \\ 0 & \text{if } v \in S_2. \end{cases}$$

In fact, since F_∞/F is a strongly admissible pro- p p -adic Lie extension, we have $F_{\infty,w} = F_w^{\mathrm{cyc}}$ when $v \notin S_2$. Therefore, in conclusion, we have

$$\mathrm{rank}_{\mathcal{O}[[H]]}(X(A/F_\infty)) = \sum_{v \notin S_2} \sum_{w|v} \mathrm{dim}_k(C_v(F_w^{\mathrm{cyc}})/\pi) + \mathrm{corank}_{k[[H]]}(\mathrm{Sel}^{str}(A[\pi]/F_\infty)) - \mathrm{corank}_{k[[H]]}(\mathrm{coker} \psi_A).$$

By a similar argument, we have the same equality for B . By the **(Cong)** assumption, we have

$$\mathrm{corank}_{k[[H]]}(\mathrm{Sel}^{str}(A[\pi]/F_\infty)) - \mathrm{corank}_{k[[H]]}(\mathrm{coker} \psi_A) = \mathrm{corank}_{k[[H]]}(\mathrm{Sel}^{str}(B[\pi]/F_\infty)) - \mathrm{corank}_{k[[H]]}(\mathrm{coker} \psi_B).$$

Combining this with the hypothesis of the proposition, we obtain

$$\mathrm{rank}_{\mathcal{O}[[H]]}(X(A/F_\infty)) = \mathrm{rank}_{\mathcal{O}[[H]]}(X(B/F_\infty))$$

as required. \square

As a corollary, we have the analog result for the G -Euler characteristics of the dual Selmer groups.

Corollary 5.4. *Retain the assumptions of Theorem 5.1. Assume further that $H_i(H, X(A/F_\infty))$ and $H_i(H, X(B/F_\infty))$ are finite for $i \geq 1$. Then we have $\chi(G, X(A/F_\infty)) = 1$ if and only if $\chi(G, X(A/F_\infty)) = 1$.*

Proof. By the hypothesis that $H_i(H, X(A/F_\infty))$ are finite for $i \geq 1$, we have that $Ak_H(X(A/F_\infty))$ is precisely the characteristic polynomial of $H_0(H, X(A/F_\infty))$. In particular, we have $Ak_H(X(A/F_\infty)) \in \mathcal{O}[[\Gamma]]$. Combining this observation with Proposition 2.3, we have $\chi(G, X(A/F_\infty)) = 1$ if and only $Ak_H(X(A/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$. Similarly, we have $\chi(G, X(B/F_\infty)) = 1$ if and only $Ak_H(X(B/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$. The conclusion of the theorem will now follow from Theorem 5.1. \square

Remark 5.5. We note that the finiteness condition on the H -homology in Corollary 5.4 is automatically satisfied if G has dimension 2 (cf. Proposition 4.3(c)).

We end the section proving a version of Theorem 5.1 for Selmer groups twisted by certain Artin representations over a strongly admissible p -adic Lie extension which is not pro- p . For the remainder of the section, we will assume that our data $(A, \{A_v\}_{v|p})$ and $(B, \{B_v\}_{v|p})$ are defined over \mathbb{Q} . Let F_∞ be a strongly admissible pro- p extension of $\mathbb{Q}(\mu_p)$ which contains $\mathbb{Q}(\mu_{p^\infty}, m^{p^{-\infty}})$ for some p -power free integer m . From now on, we write $G = \text{Gal}(F_\infty/\mathbb{Q})$, $H = \text{Gal}(F_\infty/\mathbb{Q}^{\text{cyc}})$ and $\Gamma = \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$. Note that G and H are compact p -adic group with no p -torsion but they are clearly not pro- p . For $n \geq 1$, let ρ_n denote the representation of G obtained by inducing any character of exact order p^n of $\text{Gal}(\mathbb{Q}(\mu_{p^n}, m^{p^{-n}})/\mathbb{Q}(\mu_{p^n}))$ to $\text{Gal}(\mathbb{Q}(\mu_{p^n}, m^{p^{-n}})/\mathbb{Q})$. This representation is defined over \mathbb{Q} and is irreducible of dimension $p^{n-1}(p-1)$. We may view ρ_n as an Artin representation $G \rightarrow \text{GL}_{p^{n-1}(p-1)}(\mathcal{O})$. Now if M is a $\mathcal{O}[[G]]$ -module, we define $\text{tw}_{\rho_n}(M)$ to be the \mathcal{O} -module $W_{\rho_n} \otimes_{\mathbb{Z}_p} M$ with G acting diagonally. Here W_{ρ_n} is a free \mathcal{O} -module of rank $p^{n-1}(p-1)$ which realizes ρ_n . We are in the position to state the following theorem.

Theorem 5.6. *Retain the assumptions of Theorem 5.1 and notation as above. Furthermore, assume that $X(A/F_\infty)$ and $X(B/F_\infty)$ lie in $\mathfrak{M}_H(G)$ (see Section 6 for definition). Then $Ak_H(\text{tw}_{\rho_n}X(A/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$ if and only if $Ak_H(\text{tw}_{\rho_n}X(B/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$.*

Proof. Let G_n and G'_n be the open subgroup of G corresponding to $\mathbb{Q}(\mu_{p^n}, m^{p^{-n}})$ and $\mathbb{Q}(\mu_{p^n}, m^{p^{-(n-1)}})$ respectively. Write $H_n = H \cap G_n$, $H'_n = H \cap G'_n$ and $\Gamma_n = G_n/H_n = G'_n/H'_n$. Note that G_n, G'_n, H_n and H'_n are pro- p groups without p -torsion. By [DD, Theorem A44], we have

$$Ak_{H_n}X(A/F_\infty) = Ak_{H'_n}X(A/F_\infty)N_{\Gamma/\Gamma_n}\left(Ak_H(\text{tw}_{\rho_n}X(A/F_\infty))^{p-1}\right)$$

in $Q_{\mathcal{O}}(\Gamma_n)/\mathcal{O}[[\Gamma_n]]^\times$, where $Q_{\mathcal{O}}(\Gamma_n)$ is the field of quotient of $\mathcal{O}[[\Gamma_n]]$ and N_{Γ/Γ_n} is the norm map from $\mathcal{O}[[\Gamma]]$ to $\mathcal{O}[[\Gamma_n]]$.

Now if $Ak_H(\text{tw}_{\rho_n}X(A/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$, then $N_{\Gamma/\Gamma_n}\left(Ak_H(\text{tw}_{\rho_n}X(A/F_\infty))^{p-1}\right)$ is a unit in $\mathcal{O}[[\Gamma_n]]$. Therefore, we have

$$Ak_{H_n}X(A/F_\infty) = Ak_{H'_n}X(A/F_\infty) \bmod \mathcal{O}[[\Gamma_n]]^\times.$$

Write $F_n = \mathbb{Q}(\mu_{p^n}, m^{p^{-n}})$. From now on, identify $\mathcal{O}[[\Gamma_n]] \cong \mathcal{O}[[T]]$ under a choice of a generator of Γ_n . Recall that a polynomial $T^n + c_{n-1}T^{n-1} + \cdots + c_0$ in $\mathcal{O}[T]$ is said to be a Weierstrass polynomial if π divides c_i for every $0 \leq i \leq n-1$. Since $X(A/F_\infty)$ lies in $\mathfrak{M}_H(G)$, by restricting scalars, we also have that $X(A/F_\infty)$ lies in $\mathfrak{M}_{H_n}(G_n)$. Since G_n is pro- p , we may apply [LimMu, Theorem 3.1] to conclude that

$$\mu_{\mathcal{O}[[\Gamma_n]]}(X(A/F_n^{\text{cyc}})) = \mu_{\mathcal{O}[[G_n]]}(X(A/F_\infty)).$$

By virtue of Proposition 4.3 and the above equality, we have

$$Ak_{H_n}X(A/F_\infty) = \pi^{\mu_{\mathcal{O}[[G_n]]}(X(A/F_\infty))} \frac{f}{g}$$

for some Weierstrass polynomials f and g . Similarly, we have

$$Ak_{H'_n}X(A/F_\infty) = \pi^{\mu_{\mathcal{O}[[G'_n]]}(X(A/F_\infty))} \frac{f'}{g'}$$

for some Weierstrass polynomials f' and g' . Therefore, it follows from the above Akashi series relation that $\pi^{\mu_{\mathcal{O}[[G_n]]}(X(A/F_\infty))} fg'$ and $\pi^{\mu_{\mathcal{O}[[G'_n]]}(X(A/F_\infty))} f'g$ generates the same ideal in $\mathcal{O}[[T]]$. By an \mathcal{O} -analogue of [Lim, Lemma 5.7], this in turn implies that

$$\mu_{\mathcal{O}[[G_n]]}(X(A/F_\infty)) = \mu_{\mathcal{O}[[G'_n]]}(X(A/F_\infty))$$

and

$$\deg f - \deg g = \deg f' - \deg g'.$$

On the other hand, since G' is a subgroup of G with index p , we have

$$\mu_{\mathcal{O}[[G'_n]]}(X(A/F_\infty)) = p\mu_{\mathcal{O}[[G_n]]}(X(A/F_\infty))$$

and combining this with the above, we obtain

$$\mu_{\mathcal{O}[[G_n]]}(X(A/F_\infty)) = \mu_{\mathcal{O}[[G'_n]]}(X(A/F_\infty)) = 0.$$

Since $X(A/F_\infty)$ is assumed to lie in $\mathfrak{M}_{H_n}(G_n)$, it follows from [LimMu, Proposition 5.7] that $X(A/F_\infty)$ is finitely generated over $\mathcal{O}[[H_n]]$. In particular, by the argument in the proof of [Lim, Lemma 5.4], we have

$$\text{rank}_{\mathcal{O}[[H_n]]}(X(A/F_\infty)) = \deg f - \deg g$$

Similarly, we have

$$\text{rank}_{\mathcal{O}[[H'_n]]}(X(A/F_\infty)) = \deg f' - \deg g'.$$

Since the right hand side of the two equalities are the same, we have

$$\text{rank}_{\mathcal{O}[[H_n]]}(X(A/F_\infty)) = \text{rank}_{\mathcal{O}[[H'_n]]}(X(A/F_\infty)).$$

On the other hand, since H'_n is a subgroup of H of index p , we also have

$$p \text{rank}_{\mathcal{O}[[H_n]]}(X(A/F_\infty)) = \text{rank}_{\mathcal{O}[[H'_n]]}(X(A/F_\infty)).$$

Hence we conclude that

$$\text{rank}_{\mathcal{O}[[H_n]]}(X(A/F_\infty)) = 0.$$

Therefore, we have shown that $X(A/F_\infty)$ is a finitely generated torsion $\mathcal{O}[[H_n]]$ -module. By Proposition 5.3, this in turn implies that $X(B/F_\infty)$ is also a finitely generated torsion $\mathcal{O}[[H_n]]$ -module. Since F_∞ is clearly not totally real and G_n is pro- p , we may apply Proposition 4.2 to conclude that $X(B/F_\infty) = 0$. It follows from this observation that $\text{tw}_{\rho_n}X(B/F_\infty) = 0$ which in turn implies that $Ak_H(\text{tw}_{\rho_n}X(B/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$. This completes the proof of the theorem. \square

The astute reader will notice that the proof given above can also establish the following proposition (compare with [DD, Theorem A.13]). Again, we note that we do not assume that $X(A/F_\infty)$ has no non-zero pseudo-null $\mathcal{O}[[G]]$ -submodule (but we require F_∞ to be not totally real).

Proposition 5.7. *Retain the setting of Theorem 5.6. Then $X(A/F_\infty) = 0$ if and only if there exists n such that $Ak_H(\text{tw}_{\rho_n} X(A/F_\infty))$ is a unit in $\mathcal{O}[[\Gamma]]$.*

6 Comparing characteristic elements of Selmer groups

In this section, we will prove our main results concerning with the characteristic elements of the Selmer groups. We need to introduce some further notion and notation. Let

$$\Sigma = \{ s \in \mathcal{O}[[G]] \mid \mathcal{O}[[G]]/\mathcal{O}[[G]]s \text{ is a finitely generated } \mathcal{O}[[H]]\text{-module}\}.$$

By [CFKSV, Theorem 2.4], Σ is a left and right Ore set consisting of non-zero divisors in $\mathcal{O}[[G]]$. Set $\Sigma^* = \cup_{n \geq 0} \pi^n \Sigma$. It follows from [CFKSV, Proposition 2.3] that a finitely generated $\mathcal{O}[[G]]$ -module M is annihilated by Σ^* if and only if $M/M(\pi)$ is finitely generated over $\mathcal{O}[[H]]$. We will denote $\mathfrak{M}_H(G)$ to be the category of all finitely generated $\mathcal{O}[[G]]$ -modules which are Σ^* -torsion. For data arising from abelian varieties and modular forms, it has been conjectured that the dual Selmer groups associated to these data lie in this category (see [CFKSV, CS, Su]).

By the discussion in [CFKSV, Section 3], we have the following exact sequence

$$K_1(\mathcal{O}[[G]]) \longrightarrow K_1(\mathcal{O}[[G]]_{\Sigma^*}) \xrightarrow{\partial_G} K_0(\mathfrak{M}_H(G)) \longrightarrow 0$$

of K -groups. For each M in $\mathfrak{M}_H(G)$, we define a *characteristic element* for M to be any element ξ_M in $K_1(\mathbb{Z}_p[[G]]_{\Sigma^*})$ with the property that

$$\partial_G(\xi_M) = -[M].$$

Let $\rho : G \longrightarrow GL_m(\mathcal{O}_\rho)$ denote a continuous group representation (not necessarily an Artin representation), where $\mathcal{O}' = \mathcal{O}_\rho$ is the ring of integers of some finite extension of K . For $g \in G$, we write \bar{g} for its image in $\Gamma = G/H$. We define a continuous group homomorphism

$$G \longrightarrow M_d(\mathcal{O}') \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]], \quad g \mapsto \rho(g) \otimes \bar{g}.$$

By [CFKSV, Lemma 3.3], this in turn induces a map

$$\Phi_\rho : K_1(\mathcal{O}[[G]]_{\Sigma^*}) \longrightarrow Q_{\mathcal{O}'}(\Gamma)^\times,$$

where $Q_{\mathcal{O}'}(\Gamma)$ is the field of fraction of $\mathcal{O}'[[\Gamma]]$. Let $\varphi : \mathcal{O}'[[\Gamma]] \longrightarrow \mathcal{O}'$ be the augmentation map and denote its kernel by \mathfrak{p} . One can extend φ to a map $\varphi : \mathcal{O}'[[\Gamma]]_{\mathfrak{p}} \longrightarrow K'$, where K' is the field of fraction of \mathcal{O}' . Let ξ be an arbitrary element in $K_1(\mathcal{O}[[G]]_{\Sigma^*})$. If $\Phi_\rho(\xi) \in \mathcal{O}'[[\Gamma]]_{\mathfrak{p}}$, we define $\xi(\rho)$ to be $\varphi(\Phi_\rho(\xi))$. If $\Phi_\rho(\xi) \notin \mathcal{O}'[[\Gamma]]_{\mathfrak{p}}$, we set $\xi(\rho)$ to be ∞ .

We will write α for the natural map

$$\mathcal{O}[[G]]_{\Sigma^*}^\times \longrightarrow K_1(\mathcal{O}[[G]]_{\Sigma^*}).$$

We can now state the following result which will prove [CFKSV, Conjecture 4.8 Case 4] for the characteristic elements attached to our Selmer groups. We mention that this result refines the previous result of the author [Lim, Proposition 6.3], where he proved the same result but under the extra assumption that $G = \text{Gal}(F_\infty/F)$ is pro- p .

Proposition 6.1. *Let F_∞ be a strongly admissible p -adic Lie extension of F , and assume that F_∞ is not totally real. Suppose that (\mathbf{Fin}_∞) and (\mathbf{Cong}) are satisfied. Also, suppose that $X(A/F_\infty) \in \mathfrak{M}_H(G)$. Let ξ_A be a characteristic element of $X(A/F_\infty)$. Then the following statements are equivalent.*

- (a) $\xi_A \in \alpha(\mathcal{O}[[G]]^\times)$, where α is the map $\mathcal{O}[[G]]_{\Sigma^*}^\times \longrightarrow K_1(\mathcal{O}[[G]]_{\Sigma^*})$.
- (b) $\xi_A(\rho)$ is finite and lies in \mathcal{O}_ρ^\times for every continuous group representation ρ of G .
- (c) $\Phi_\rho(\xi_A) \in \mathcal{O}_\rho[[\Gamma]]^\times$ for every continuous group representation ρ of G .
- (d) $\Phi_\rho(\xi_A) \in \mathcal{O}_\rho[[\Gamma]]^\times$ for every Artin representation ρ of G .
- (e) There exists an open normal pro- p subgroup G' of G such $Ak_{H'}(X(A/F_\infty)) \in \mathcal{O}[[\Gamma']]^\times$. Here $H' = H \cap G'$ and $\Gamma' = G'/H'$.

Proof. By [CFKSV, Lemma 4.9], we have the implications (a) \Rightarrow (b) \Leftrightarrow (c) \Rightarrow (d). It remains to show (d) \Rightarrow (e) \Rightarrow (a). We first establish the implication (d) \Rightarrow (e). Let G' be an open normal subgroup of G with the property that G' is pro- p . Write $H' = H \cap G'$ and $\Gamma' = G'/H'$. Let \mathcal{O}' denote the ring of integers of a finite extension K' of \mathbb{Q}_p such that K' contains K and all absolutely irreducible representations of $\Delta = G/G'$ can be realized over K' . Denote $\hat{\Delta}$ to be the set of all irreducible representations of Δ . Write $M_{\mathcal{O}'} = M \otimes_{\mathcal{O}} \mathcal{O}'$. If M is a $\mathcal{O}[[G]]$ -module and $\rho \in \hat{\Delta}$ with dimension n_ρ , we define $\text{tw}_\rho(M)$ to be the \mathcal{O}' -module $W_\rho \otimes_{\mathcal{O}'} M_{\mathcal{O}'}$ with G acting diagonally. Here W_ρ is a free \mathcal{O}' -module of rank n_ρ realizing ρ . Then by [DD, Theorem A44], we have

$$Ak_{H'}(X(A/F_\infty)_{\mathcal{O}'})^{[\Gamma:\Gamma']} = N_{\Gamma/\Gamma'} \left(\prod_{\rho \in \hat{\Delta}} Ak_H(\text{tw}_\rho X(A/F_\infty))^{n_\rho} \right) \bmod \mathcal{O}'[[\Gamma]]^\times,$$

where $N_{\Gamma/\Gamma'}$ is the norm map from $\mathcal{O}[[\Gamma]]$ to $\mathcal{O}[[\Gamma']]$ and n_ρ is the dimension of ρ . On the other hand, it follows from [CFKSV, Lemma 3.7] that one has

$$\Phi_\rho(\xi_A) = Ak_H(\text{tw}_{\hat{\rho}} X(A/F_\infty)) \bmod \mathcal{O}'[[\Gamma]]^\times,$$

where $\hat{\rho}$ is the contragredient of ρ . Now if statement (d) holds, it then follows from combining the above two observations that

$$Ak_{H'}(X(A/F_\infty)_{\mathcal{O}'}) \in \mathcal{O}'[[\Gamma]]^\times.$$

It is now an easy exercise to deduce from this that

$$Ak_{H'}(X(A/F_\infty)) \in \mathcal{O}[[\Gamma]]^\times.$$

This proves the implication (d) \Rightarrow (e)

We now prove (e) \Rightarrow (a). Suppose that statement (e) holds. Let F' be the fixed field of G' as given by (e). Since we are assuming that $X(A/F_\infty) \in \mathfrak{M}_H(G)$, we may apply [CS, Proposition 2.5] to conclude that

$X(A/L^{\text{cyc}})$ is $\mathcal{O}[[\Gamma_L]]$ -torsion for every finite extension L of F contained in F_∞ , where $\Gamma_L = \text{Gal}(L^{\text{cyc}}/L)$. Combining this with our (\mathbf{Fin}_∞) assumption, it follows from a similar argument to that in [Lim, Proposition 3.3(i), Corollary 3.4] that $H^2(G_S(F_\infty), A) = 0$ and λ_{A/F_∞} is surjective. Therefore, we may apply Propositions 2.2 and 4.3 to conclude that $X(A/F_\infty)$ is a finitely generated torsion $\mathcal{O}[[H']]$ -module. Since F_∞ is not totally real and G' is pro- p , it follows from Lemma 4.2(a) that $X(A/F_\infty) = 0$. In particular, this implies that $\partial_G(\xi_A) = 0$. Consequently, it follows from the exact sequence of K -groups

$$K_1(\mathcal{O}[[G]]) \longrightarrow K_1(\mathcal{O}[[G]]_{\Sigma^*}) \xrightarrow{\partial_G} K_0(\mathfrak{M}_H(G)) \longrightarrow 0$$

that there exists an element in $K_1(\mathcal{O}[[G]])$ which maps to ξ_A . On the other hand, since $\mathcal{O}[[G]]$ is semi-local, we have that $\mathcal{O}[[G]]^\times$ maps onto $K_1(\mathcal{O}[[G]])$, thus yielding (a). \square

We mention that one can also establish the following proposition which will refine [Lim, Proposition 6.2]. As the proof is very similar to the preceding proposition, we will omit it.

Proposition 6.2. *Let $M \in \mathfrak{M}_H(G)$. Suppose that M contain no nontrivial pseudo-null $\mathcal{O}[[G]]$ -submodules. Let ξ_M be a characteristic element of M . Then the following statements are equivalent.*

- (a) $\xi_M \in \alpha(\mathcal{O}[[G]]^\times)$, where α is the map $\mathcal{O}[[G]]_{\Sigma^*}^\times \longrightarrow K_1(\mathcal{O}[[G]]_{\Sigma^*})$.
- (b) $\xi_M(\rho)$ is finite and lies in \mathcal{O}_ρ^\times for every continuous group representation ρ of G .
- (c) $\Phi_\rho(\xi_M) \in \mathcal{O}_\rho[[\Gamma]]^\times$ for every continuous group representation ρ of G .
- (d) $\Phi_\rho(\xi_M) \in \mathcal{O}_\rho[[\Gamma]]^\times$ for every Artin representation ρ of G .

We can now prove our final result which compares the characteristic elements of the Selmer groups of two congruent data.

Theorem 6.3. *Let F_∞ be a strongly admissible p -adic Lie extension of F . Assume that F is not totally real. Suppose that all the hypothesis in Proposition 4.3 are valid for A and B , and that (\mathbf{Cong}) is satisfied. Furthermore, suppose that for every $v \notin S_2$, we have*

$$\dim_k (C_v(F_w^{\text{cyc}})/\pi) = \dim_k (D_v(F_w^{\text{cyc}})/\pi)$$

for every prime w of F^{cyc} above v .

Then $\xi_A \in \alpha(\mathcal{O}[[G]]^\times)$ if and only if $\xi_B \in \alpha(\mathcal{O}[[G]]^\times)$.

Proof. Suppose that $\xi_A \in \alpha(\mathcal{O}[[G]]^\times)$. Then by Proposition 6.1, there exists an open normal pro- p subgroup G' of G such $Ak_{H'}(X(A/F_\infty)) \in \mathcal{O}[[\Gamma]]^\times$, where $H' = H \cap G'$. By Theorem 5.1, this in turn implies that $Ak_{H'}(X(B/F_\infty)) \in \mathcal{O}[[\Gamma]]^\times$. We may now apply Proposition 6.1 again to conclude that $\xi_B \in \alpha(\mathcal{O}[[G]]^\times)$. \square

Acknowledgments. We like to thank the anonymous referee for pointing out some mistakes and valuable comments which improved the clarity of the paper.

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